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# ON THE APPLICATION OF THEORY OF MARKOV PROCESSES TO THE EVALUATION OF STATE OF DYNAMIC SYSTEMS AND TO CONTROL OF AIRCRAFT OSCILLATIONS

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16. Abstract

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# ON THE APPLICATION OF THE THEORY OF MARKOV PROCESSES TO EVALUATION OF STATE OF DYNAMIC SYSTEMS AND TO CONTROL OF AIRCRAFT OSCILLATIONS

V. T. Tarushkin

The application of the theory of Markov processes and stochastic differential equations to the evaluation of state of linear dynamic systems and to control of small oscillations of an aircraft in circular orbit is discussed. /150

## 1. Some Notions of the Theory of Stochastic Differential Equations

Given that $x_t$ is an n-dimensional Markov process satisfying a system of stochastic differential equations in the Ito form [1], i.e.,

$$dx_t = f(x_t, t)\,dt + \sigma(x_t, t)\,dw(t). \tag{1.1}$$

Here, $f(x_t, t)$ is an n-dimensional vector of transposition, $\sigma(x_t, t)$ /151 is a diffusion matrix of dimensionality n X m, w(t) is an m-dimensional process of Brownian movement with independent components and unique dispersion parameter. Equation (1.1) is equivalent to the integral equation

$$x_t = x_0 + \int_{t_0}^{t} f(x_\tau, \tau)\,d\tau + \int_{t_0}^{t} \sigma(x_\tau, \tau)\,dw(\tau). \tag{1.2}$$

The last integral in (1.2) is stochastic and is defined as the limit in the average quadratic of integral sums, i.e.,

$$\int_{t_0}^{t} \sigma(x_t, t)\,dw = \underset{\tau\to 0}{\mathrm{l.\,i.\,m.}} \sum_{i=1}^{N} \sigma(x_{t_i}, t_i)[w(t_{i+1}) - w(t_i)], \tag{1.3}$$

where $\tau = \max_i (t_{i+1} = t_i)$.

By analogy to (1.1), we can define the stochastic differential equation in the Stratonovich form [2], i.e.,

$$d^*x_t = f(x_t, t)\,dt + \sigma(x_t, t)\,d^*w. \tag{1.4}$$

Equation (1.4) is equivalent to the equation

$$x_t = x_0 + \int_{t_0}^{t} f(x_\tau, \tau)\,d\tau + \int_{t_0}^{t} \sigma(x_\tau, \tau)\,d^*w. \tag{1.5}$$

The last integral in (1.5) is balanced stochastic integral and is defined as the limit in the average quadratic of the integral sums of the following form:

$$\int_{t_0}^{t} \sigma(x_t, t)\,d^*w = \underset{\tau \to 0}{\mathrm{l.\,i.\,m.}} \sum_{i=1}^{N} \sigma\left(\frac{x_{t_i} + x_{t_{i+1}}}{2}, t_i\right)[w(t_{i+1}) - w(t_i)]. \tag{1.6}$$

Let us note that if $\sigma(x_t, t)$ is not a function of $x_t$, definitions for both integrals coincide. Below, this case will be examined.

The ordinarily examined dynamic system are characterized by the fact that their trajectories virtually everywhere are differentiable. The process of Brownian movement, used in equations (1.1)-(1.4) is not differentiable in the ordinary sense [3]. This circumstance proves that we should, in a certain sense, define a generalized derivative of the process of Brownian movement if we want to have a convenient mathematical apparatus. The case of indifferentiability of the probability density is usually eliminated by introducing generalized $\delta$-functions of Dirac [4]. Below we will define the derivative v of the process of Brownian movement as a generalized Gaussian m-dimensional random process of the type of white noise, having a zero average

$$Ev = 0 \tag{1.7}$$ /152

and a covariation matrix

$$Ev(t)\,v(s)^{\tau} = R(t)\,\delta(t-s). \tag{1.8}$$

Here, the exponent "t" denotes transposition, while R is the matrix of dimensionality m X m.

Let us introduce the process

$$\nu(t) = \int_{t_0}^{t} \sigma(\tau)\,dw(\tau). \tag{1.9}$$

Similarly, for s > t we will define that

$$\nu(s) = \int_{t}^{s} \sigma(\tau)\,dw(\tau).$$

In virtue of the fact that increase in Brownian movement have a zero average, we find that

$$E\nu(t) = E\nu(s) = 0. \tag{1.10}$$

We will further find that

$$E\nu(t)\,\nu(t)^{\tau} = \int_{t_0}^{t} \sigma(\tau)\,\sigma(\tau)^{\tau}\,d\tau, \tag{1.11}$$

$$E\nu(t)\,\nu(s)^{\tau} = 0. \tag{1.12}$$

To prove equation (1.10), we only have to apply the definition of the stochastic integral to (1.9) and notes that for each partial sum

$$ES_N = \sum_{i=1}^{N} \sigma(\tau_i)\,E\{w(\tau_{i+1}) - w(\tau_i)\} = 0.$$

To prove (1.11), we only have to apply the definition of the stochastic integral and write the product of partial sums

$$ES_N S_M = \sum_{i,j=1}^{N,M} \sigma(\tau_i) E\{w(\tau_{i+1}) - w(\tau_i)\} \times$$
$$\times \{w(\tau_{j+1}) - w(\tau_j)\}^{\tau} \sigma(\tau_j)^{\tau} =$$
$$= \sum_{i=1}^{N} \sigma(\tau_i) \sigma(\tau_i)^{\tau} (\tau_{i+1} - \tau_i).$$

The latter relationship takes place because dispersions of increases in Brownian movement are equal to the differences between the initial and final moments in time, while increases for nonintersecting intervals of time are independent.

By analogy equation (1.12) is proven, but in this case we should note that all intervals of division for partial sums are nonintersecting.

Let us introduce the derivative v with the aid of an equation similar to (1.9), i.e.,

$$\int_{t_0}^{t} \sigma(\tau)\, dw(\tau) = \int_{t_0}^{t} v\, d\tau. \tag{1.13}$$

The properties of (1.7)-(1.8), which this derivative possesses, are simply a convenient form of writing relations (1.10)-(1.12) using generalized functions.

Thus, we will finally define the stochastic differential equation in the form

$$\frac{dx_t}{dt} = f(x_t, t) + v. \tag{1.14}$$

Equation (1.14) is equivalent to the integral equation

$$x_t = x_0 + \int_{t_0}^{t} f(x_\tau, \tau)\, d\tau + \int_{t_0}^{t} v\, d\tau. \tag{1.15}$$

The last integral in (1.15) is an analog of the stochastic integral and is defined by equation (1.13) using the introduced function v, which has the sense of a generalized derivative of the process of Brownian movement.

Equations (1.10)-(1.12), with the aid of function v, will be written as

$$E \int_{t_0}^{t} v\, d\tau = E \int_{t}^{s} v\, d\tau = 0, \tag{1.16}$$

$$E\left\{\int_{t_0}^{t} v(\tau)\, d\tau\right\}\left\{\int_{t_0}^{t} v(s)\, ds\right\}^{\tau} = \int_{t_0}^{t} R(\tau)\, d\tau, \tag{1.17}$$

$$E\left\{\int_{t_0}^{t} v(\tau)\, d\tau\right\}\left\{\int_{t}^{s} v(p)\, dp\right\}^{\tau} = 0. \tag{1.18}$$

These relations are proven using δ-functions, whereas relations (1.10)-(1.12) are proven using Ito-Stratonovich integrals. From (1.17) it follows that

$$R = \sigma\sigma^{\tau}. \tag{1.19}$$

Let us note that (t) and (s) are defined above as values of the integral (1.9) for nonintersecting intervals of time and have the sense of discrete values of the process with independent values, definable by the integral of the generalized derivative of the process of Brownian movement. Positing that $t = t_i$,

$$s = t_j, \quad \nu(t_i) = \nu_i, \quad \nu(t_j) = \nu_j, \quad \int_{t_0}^{t_i} R\, d\tau = R_i,$$

we find that

$$E\nu_i = E\nu_j = 0, \quad E\nu_i\nu_j^T = R_i\delta_{ij}, \tag{1.20}$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

For small intervals of integration, integral (1.9) can be written /154
as

$$\nu(t) = \sigma(t_k)\left[w(t) - w(t_0)\right]. \tag{1.21}$$

Here $t_k \in [t, t_0]$. Since for each $i^{th}$ component of Brownian movement there takes place the inequality

$$\text{Вер}\{w_i(t) - w_i(t_0) < x_i\} = \frac{1}{\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{x_i} e^{-\frac{z_i^2}{2(t-t_0)}} dz_i, \tag{1.22}$$

then (1.21) can be seen as a linear transformation of normally distributed, independent values. Accordingly, for small intervals of integration, the discrete process defined by (1.20) belongs to the normal law with a mathematical expectation equal to zero and covariation matrix $R_i$.

## 2. Evaluation of State in Partially Observable Linear Systems

Given that $x_t$ is an n-dimensional unobservable, while $y_t$ is an m-dimensional observable of the component of a Markov random process which satisfies a system of stochastic differential equations in the Ito-Stratonovich form:

$$dx_t = A(t)x_t\,dt + C(t)\,dw_1(t), \tag{2.1}$$

$$dy_t = H(t)\, x_t\, dt + G(t)\, dw_2(t). \tag{2.2}$$

Here, A, C, H, G are matrices of dimensionalities n X n, n X p, m X n, n Q q; $w_1$ is p-dimensional, while $w_2$ is q-dimensional--process of Brownian movement having independent components and unique dispersion parameters. For system (2.1)-(2.2), we assume the fulfillment of the condition of theorem 1' from [5]. Then, if the a priori distribution for system (2.1)-(2.2) is Gaussian with the parameters

$$\hat{x}_0 = Ex_0, \quad P_0 = E(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^\tau,$$

then the a posteriori distribution is Gaussian with the parameters

$$\hat{x}_t = E\{x_t / Y_t\}, \quad P = E\{(x_t - \hat{x}_t)(x_t - \hat{x}_t)^\tau / Y_t\},$$

satisfying the equations

$$d\hat{x}_t = A\hat{x}_t\, dt + PH^\tau [GG^\tau]^{-1} [dy_t - H\hat{x}_t\, dt]. \tag{2.3}$$

$$\frac{dP}{dt} = AP + PA^\tau + CC^\tau - PH^\tau [GG^\tau]^{-1} HP. \tag{2.4}$$

Here $Y_t = \sigma\{y_s,\ 0 \leq s \leq t\}$-- σ-algebra generated by the values of $y_s$ for $0 \leq s \leq t$. Let us note that for integration of equation (2.4), the knowledge of realization is not required.

Let us introduce the generalized derivatives /155

$$w = C\frac{dw_1}{dt}, \quad v = G\frac{dw_2}{dt},$$

Then, instead of system (2.1)-(2.2), we will have

$$\frac{dx_t}{dt} = Ax_t + w, \tag{2.5}$$

$$z_t = Hx_t + v, \quad z_t = \frac{dy_t}{dt}. \tag{2.6}$$

Here, v and w are Gaussian processes of white noise, defined in the preceeding section and having zero averages

$$Ew = Ev = 0 \tag{2.7}$$

and covariation matrices

$$\begin{aligned} Ew(t)\, w(t+s)^{\tau} &= Q(t)\, \delta(t-s), \\ Ev(t)\, v(t+s)^{\tau} &= R(t)\, \delta(t-s), \end{aligned} \tag{2.8}$$

where in conformity with (1.19)

$$R = GG^{\tau}, \quad Q = CC^{\tau}. \tag{2.9}$$

System (2.5) is equivalent to the integral equation in the interval $[t_0, t]$:

$$x_t = x_0 + \int_{t_0}^{t} Ax_{\tau}\, d\tau + \int_{t_0}^{t} w\, d\tau. \tag{2.10}$$

Let us designate by $\Phi(k = 1, k) = \Phi(t_{k+1}, t_k)$ the fundamental matrix of the system

$$\frac{dx_t}{dt} = Ax_t$$

such that $\Phi(t_k, t_{k+1}) = I$, where I is a unique matrix. Equation (2.10) can not be written in the interval $[t_k, t_{k+1}]$ in the form

$$x_{k+1} = \Phi(k+1, k)\, x_k + w_k. \tag{2.11}$$

Here it is designated that $x_{k+1} = x(t_{k+1}),\ x_k = x(t_k),\quad w_k = \int_{t_k}^{t_{k+1}} w\, d\tau.$

The procedure of transition from equation (2.5) to equation (2.11) will be called discretization. By selecting the intervals of discretization so that they do not intersect, we find that on the basis of (1.20) for any $[t_k, t_{k+1}]$, $[t_i, t_{i+1}]$

$$Ew_i = Ew_k = 0, \quad Ew_i w_k^T = Q_i \delta_{ik}. \tag{2.12}$$

System (2.6) is equivalent to the integral equation /156

$$\frac{y_t - y_0}{t - t_0} = \frac{1}{t - t_0} \int_{t_0}^{t} Hx_\tau \, d\tau + \frac{1}{t - t_0} \int_{t_0}^{t} v \, d\tau. \tag{2.13}$$

For the interval of discretization $[t_k, t_{k+1}]$, equation (2.13) is written as

$$z_k = H_k x_k + v_k. \tag{2.14}$$

Here it is designated that

$$z_k = \frac{y_{k+1} - y_k}{t_{k+1} - t_k}, \quad H_k x_k = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} Hx_\tau \, d\tau,$$
$$v_k = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} v \, d\tau. \tag{2.15}$$

For any $[t_k, t_{k+1}]$, $[t_i, t_{i+1}]$, according to (1.20), we find that

$$Ev_k = Ev_i = 0, \quad Ev_k v_i^T = R_i \delta_{ik}. \tag{2.16}$$

We must also take into account that

$$R_i = \frac{1}{(t_{i+1}-t_i)^2} \int_{t_i}^{t_{i+1}} R \, d\tau. \tag{2.17}$$

For system (2.11)-(2.14) and discrete white noise possessing properties (2.12)-(2.16), the estimation of the state by the method of maximal probability is derived in [6]; the interrelationship of different forms of estimation is discussed in [7].

Estimation of state is defined by the equations

$$\begin{gathered} \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - H_k \hat{x}_{k|k-1}), \\ K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}, \\ P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1}. \end{gathered} \tag{2.18}$$

Here it is designated that

$$\begin{gathered} \hat{x}_{k-1|k-1} = E(x_{k-1}/Y_{k-1}), \quad \hat{x}_{k|k-1} = E(x_k/Y_{k-1}), \\ \tilde{x}_{k-1|k-1} = x_{k-1} - \hat{x}_{k-1|k-1}, \quad \tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}, \\ P_{k-1|k-1} = E\tilde{x}_{k-1|k-1}\tilde{x}^T_{k-1|k-1}, \\ P_{k|k-1} = E\tilde{x}_{k|k-1}\tilde{x}^T_{k|k-1}, \\ Y_{k-1} = \{z_1, \ldots, z_{k-1}\}. \end{gathered}$$

In addition to the form of (2.18), estimation of state can be written as

$$\begin{gathered} \hat{x}_{k|k} = x_{k|k-1} + K_k (z_k - H_k \hat{x}_{k|k-1}), \\ K_k = P_{k|k} H_k^T R_k^{-1}, \\ P_{k|k}^{-1} = P_{k|k-1}^{-1} + H_k^T R_k^{-1} H_k. \end{gathered} \tag{2.19}$$

We will show that the estimation derived by (2.18), (2.19) is unbiased, if the zero approximation is unbiased. In virtue of the /157 fact that

$$\hat{x}_{k|k-1}=\Phi(k, k-1)\hat{x}_{k-1|k-1}, \quad Ez_k=H_kEx_k,$$

we find from (2.18), (2.19), taking the mathematical expectation, that

$$E\hat{x}_{k|k}=\Phi(k, k-1)E\hat{x}_{k-1|k-1}+K_kH_k\Phi(k, k-1)[Ex_{k-1}-Ex_{k-1|k-1}].$$

Hence in virtue of the fact that

$$Ex_k=\Phi(k, k-1)Ex_{k-1}, \quad E(x_0-\hat{x}_0)=0,$$

by the method of mathematical induction we find the absence of bias of estimate of

$$E(x_k-\hat{x}_k)=0.$$

In [6] it is shown that as the interval of discretization approaches zero, we find for the discrete form of estimates their continuous analog

$$\frac{d\hat{x}_t}{dt}=Ax_t+K[z-H\hat{x}_t], \quad K=PH^{\mathsf{T}}R^{-1}, \tag{2.20}$$

$$\frac{dP}{dt}=AP+PA^{\mathsf{T}}-PH^{\mathsf{T}}R^{-1}HP+Q. \tag{2.21}$$

We will show that relationship (2.20)-(2.21) is a generalized form of equations (2.3)-(2.4); we will likewise examine those circumstances at which the solution of (2.3)-(2.4) is itself a Markov diffusion process.

Let us introduce the process

$$d\xi_t = dy_t - H\hat{x}_t\,dt. \tag{2.22}$$

Integrating (2.22) with allowance for (2.2), we find that

$$\xi_{t_2} = \xi_{t_1} + \int_{t_1}^{t_2} H[x_t - \hat{x}_t]\,dt + \int_{t_1}^{t_2} G\,dw_2. \tag{2.23}$$

By virtue of the properties of the stochastic integral [1], we find that

$$E\left\{\int_{t_1}^{t_2} G\,dw_2/Y_{t_1}\right\} = 0. \tag{2.24}$$

Hence, by applying the operation of arbitrary mathematical expectation to (2.22), we will find that

$$E\{\xi_{t_2}/Y_{t_1}\} = \xi_{t_1} + \int_{t_1}^{t_2} HE\{(x_t - \hat{x}_t)/Y_{t_1}\}\,dt. \tag{2.25}$$

Since $Y_{t1} \subset Y_t$ for $t \in [t_1, t_2]$ and with a probability of 1, by /158
virtue of the properties of reiterated arbitrary probabilities [3], we find that

$$E\{(x_t - \hat{x}_t)/Y_{t_1}\} = E\{E\{(x_t - \hat{x}_t)/Y_t\}/Y_{t_1}\},$$

then from (2.25) we find that $E\{\xi_{t2}/Y_{t1}\} = \xi_{t1}$, i.e., the process $\xi_t$ is a martingal. Consequently, the circumstances of theorem 5.3 [3] (page 403) define the conditions at which the following equation takes place

$$d\xi_t = \Phi(y_t, t)\,du(t), \tag{2.26}$$

where u(t) is a process of Brownian movement, while $\Phi(y_t, t)$ is some matrix.

The conditions of the theorem require that the following takes place with a probability of 1

$$E\{[\xi_{t_2}-\xi_{t_1}][\xi_{t_2}-\xi_{t_1}]^{\mathrm{T}}/Y_{t_1}\}=E\left\{\int_{t_1}^{t_2}\Phi^2(y_t,t)\,dt/Y_{t_1}\right\}. \tag{2.27}$$

Relationship (2.27) signifies that the process $\xi_t$ must be a D-martingal as well [8]; consequently, it follows that the process $\xi_t$ itself is analogous to the process of Brownian movement in its use for constructing a stochastic integral. This yields a second method of constructing the theory which differs from the use of theorem 5.3 [3]. For small intervals $[t_1, t_2]$, we will calculate the left side of relationship (2.27). We find that

$$\begin{aligned}
&\xi_{t_2}-\xi_{t_1}=\int_{t_1}^{t_2}H[x_t-\hat{x}_t]\,dt+\int_{t_1}^{t_2}G\,dw_2,\\
&[\xi_{t_2}-\xi_{t_1}][\xi_{t_2}-\xi_{t_1}]^{\mathrm{T}}=\left[\int_{t_1}^{t_2}H[x_t-\hat{x}_t]\,dt+\int_{t_1}^{t_2}G\,dw_2\right]\times\\
&\quad\times\left[\int_{t_1}^{t_2}[x_t-\hat{x}_t]^{\mathrm{T}}H^{\mathrm{T}}\,dt+\int_{t_1}^{t_2}G^{\mathrm{T}}\,dw_2\right]=\\
&\quad=\left[\int_{t_1}^{t_2}H[x_t-\hat{x}_t]\,dt\right]\left[\int_{t_1}^{t_2}[x_t-\hat{x}_t]^{\mathrm{T}}H^{\mathrm{T}}\,dt\right]+\\
&\quad+\left[\int_{t_1}^{t_2}G\,dw_2\right]\left[\int_{t_1}^{t_2}[x_t-\hat{x}_t]^{\mathrm{T}}H^{\mathrm{T}}\,dt\right]+\\
&\quad+\left[\int_{t_1}^{t_2}H[x_t-\hat{x}_t]\,dt\right]\left[\int_{t_1}^{t_2}G^{\mathrm{T}}\,dw_2\right]+\\
&\quad+\left[\int_{t_1}^{t_2}G\,dw_2\right]\left[\int_{t_1}^{t_2}G^{\mathrm{T}}\,dw_2\right].
\end{aligned}$$

It is easy to see that the first term in (2.28) is a quantity of the second order of smallness for small $[t_1, t_2]$. If we apply the operation of arbitrary mathematical expectation, then the second and third terms will also be zeroes by virtue of the /159 property of the stochastic integral, analogous to (2.24).

Therefore, we find that

$$E\{[\xi_{t_2}-\xi_{t_1}][\xi_{t_2}-\xi_{t_1}]^T/Y_{t_1}\}= = E\left\{\left[\int_{t_1}^{t_2} G\,dw_2\right]\left[\int_{t_1}^{t_2} G^T dw_2\right]\Big/ Y_{t_1}\right\}. \quad (2.29)$$

By virtue of the properties of the stochastic integral [1]

$$E\left\{\left[\int_{t_1}^{t_2} G\,dw_2\right]\left[\int_{t_1}^{t_2} G^T dw_2\right]\Big/ Y_{t_1}\right\}=\int_{t_1}^{t_2} GG^T dt. \quad (2.30)$$

From (2.29)-(2.30) it follows that $\Phi = [GG^T]^{1/2}$ is also a determinate function in this case. Thus, equation (2.3) can be written in the following forms:

$$d\hat{x}_t = A\hat{x}_t\,dt + PH^T[GG^T]^{-1/2}\,du, \quad (2.31)$$

$$d\hat{x}_t = A\hat{x}_t\,dt + PH^T[GG^T]^{-1}\,d\xi_t. \quad (2.32)$$

Equation (2.31) corresponds to the Ito-Stratonovich equation for a diffused Markov process, while equation (2.32) is also stochastic, but the differential is taken in terms of a martingal. Both equations will be equivalent to each other if the matrix $GG^T$ is positively defined. Since the generalized derivative of the process of Brownian movement is defined, then in accordance with (2.26) the generalized derivative of the D-martingal $\xi_t$ will be defined, while from (2.22) we find that

$$\frac{d\xi_t}{dt}=\frac{dy_t}{dt}-H\hat{x}_t=z_t-H\hat{x}_t.$$

Thus, equation (2.20) should be seen as a stochastic differential equation which contains a generalized differential of the martingal.

Let us investigate the basis of the limiting process from (2.18)-(2.19) to (2.20)-(2.21) in the interval $[t_0, t]$ which in the process of discretization was broken into partial intervals $[t_k, t_{k+1}]$. For each $[t_k, t_{k+1}] \subset [t_0, t_1]$ we find that

$$E(\hat{x}_{k|k}/Y_{k-1}) = E(\hat{x}_{k|k-1}/Y_{k-1}) + \\ + K_k \{H_k E(x_k/Y_{k-1}) - H_k E(\hat{x}_{k|k-1}/Y_{k-1})\} + E(v_k/Y_{k-1}). \tag{2.33}$$

Since interference $v_k = z_k - H_k x_k$ is not a function of $Y_{k-1} = \{z_1, \ldots, z_{k-1}\}$, then $E\{v_k/Y_{k-1}\} = Ev_k = 0$, due to the fact that $E\{x_k/Y_{k-1}\} = x_{k/k-1}$ and with a probability of 1 $E\{\hat{x}_{k/k-1}/Y_{k-1}\} = \hat{x}_{k/k-1}$, if $E\{\hat{x}_{k/k-1}\} < \infty$, then with a probability of 1

$$H_k E(x_k/Y_{k-1}) - H_k E(\hat{x}_{k|k-1}/Y_{k-1}) = 0.$$

and (2.33) acquires the form /160

$$E\{\hat{x}_{k|k}/Y_{k-1}\} = \hat{x}_{k|k-1}. \tag{2.34}$$

Let us introduce the rank of fine subdivision $\lambda = \max_k (t_{k+1} - t_k)$. Since $\lim_{\lambda \to 0} \hat{x}_{k/k-1} = x_{k-1/k-1}$, then in the limit we find that

$$E\{\hat{x}_{k|k}/Y_{k-1}\} = \hat{x}_{k-1|k-1},$$

i.e., the sequence $\{\hat{x}_{k/k}, Y_k, k = 1,2,\ldots\}$ is a martingal; then, in conformity with theorem 4.1 [3] (p. 287), if $\lim_{k\to\infty} E\{|\hat{x}_{k/k}|\} < \infty$, then with a probability of 1 exists $\lim_{k\to\infty} x_{k/k}$.

## 3. Construction of an Estimate Using Stationary Linear Systems

Given in the interval $[t_0, t_1]$ we must numerically find a solution of equations (2.1)-(2.2) which satisfies the initial conditions

$$\hat{x}_0 = \hat{x}_t|_{t=t_0}, \quad P_0 = P(t_0).$$

Let us subdivide this interval into points of subdivision

$$t_0 = \tau_0 < \tau_1 < \ldots < \tau_n = t_1.$$

In each interval of subdivision $(\tau_N, \tau_{N+1})$, let us select the points

$$\tau_N \leq t_0^N < t_1^N < \ldots < t_q^N < \tau_{N+1}.$$

Let us introduce

$$z_N = \begin{pmatrix} z(t_0^N) \\ z(t_1^N) \\ \vdots \\ z(t_q^N) \end{pmatrix}, \quad H_N = \begin{pmatrix} H(t_0^N) \\ H(t_1^N) \\ \vdots \\ H(t_q^N) \end{pmatrix}, \quad v_N = \begin{pmatrix} v(t_0^N) \\ v(t_1^N) \\ \vdots \\ v(t_q^N) \end{pmatrix} \tag{3.1}$$

and form the system

$$z_N = H_N x_N + v_N \quad (N = 0, 1, \ldots, n-1). \tag{3.2}$$

Here, by virtue of the fact that

$$Ev(t_i^N) = Ev(t_j^N) = 0, \quad Ev(t_i^N)\, v(t_j^N)^{\tau} = R_i \delta_{ij},$$

in the unified system (3.2) will appear

$$Ev_N = Ev_M = 0, \quad Ev_N v_M^T = R_N \delta_{NM}, \tag{3.3}$$

where

$$R_N = \begin{pmatrix} R_0^N & 0 & \dots & 0 \\ 0 & R_1^N & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & R_N \end{pmatrix}.$$

By definition, the system is called stationary if its state in each interval of subdivision does not change, i.e., /161

$$x_N = x(t_0^N) = x(t_1^N) = \dots = x(t_q^N). \tag{3.4}$$

By virtue of (3.4), discretization of equation (2.10) can be fulfilled at each $[\tau_N, \tau_{N+1}]$ and, instead of system (2.11), we derive the system

$$x_{N+1} = \Phi(N+1, N)\, x_N + w_N, \tag{3.5}$$

where as before we find that

$$Ew_N = Ew_M = 0, \quad Ew_N w_M^T = Q_N \delta_{NM}. \tag{3.6}$$

Here is designated that

$$\begin{gathered} x_{N+1} = x(t), \quad t \in [\tau_{N+1}, \tau_{N+2}], \\ x_N = x(t), \quad t \in [\tau_N, \tau_{N+1}], \\ \Phi(N+1, N) = \Phi(t, \tau), \quad t \in [\tau_{N+1}, \tau_{N+2}], \\ \tau \in [\tau_N, \tau_{N+1}]. \end{gathered}$$

Therefore, the discrete system (3.2)-(3.3), (3.5)-(3.6) coincides in form with system (2.11)-(2.12), (2.14)-(2.16); the estimates (2.18), (2.19) are valid for it only with the substitution of the exponent N for the exponent k.

## 4. The Control of Small Oscillations of an Aircraft in a Circular Orbit

In conformity with [9-10], let us examine the problem of control of small forced oscillations of a device in circular orbit in the central field, in the presence of measurements of one reference vector. Motion is examined in terms of an orbital system of coordinates $x_0y_0z_0$ ($x_0$--in direction of positive transversal; $z_0$--in direction of radius-vector; $y_0$ forms a directed trihedral with the first two). The system of coordinates xyz is rigidly attached to the body. For any moment in time we find the following relation:

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} H_{x_0} \\ H_{y_0} \\ H_{z_0} \end{pmatrix} + v. \tag{4.1}$$

Here $a_{ij}$ are transformation matrix elements which are expressed in some manner by the angles of of pitch , yaw , and bank ; ($H_x$, $H_y$, $H_z$) are projections of the reference vector in a connected system measurable by sensor; ($H_{x0}$, $H_{y0}$, $H_{z0}$) are projections of the reference vector in an orbital system known from theoretical formulas; $v = \{v_x, v_y, v_z\}^T$--the vector of additive interference of /162
the white noise type, having a zero average and covariation matrix

$$Ev(t)v(s)^T = R(t)\delta(t-s).$$

In addition to the matrix R, we will use the matrix G connected to R by the equation $R = GG^T$. Relation (4.1), in the absence of interference, is an ordinary transformation of coordinates. For forced oscillations, pitch, bank and yaw are predicted in the form of fragments of the Fourier series:

$$\begin{aligned}\psi&=d_1+d_2\sin\omega t+d_3\cos\omega t+d_4\sin 2\omega t+d_5\cos 2\omega t,\\ \theta&=d_6+d_7\sin\omega t+d_8\cos\omega t+d_9\sin 2\omega t+d_{10}\cos 2\omega t,\\ \varphi&=d_{11}+d_{12}\sin\omega t+d_{13}\cos\omega t+\omega_{14}\sin 2\omega t+d_{15}\cos 2\omega t,\end{aligned}$$

where $\omega$ = const is a known frequency.

The estimation of state is effected in the interval $[t_0, t_1]$, which is broken by subdivision points into intervals $[\tau_N, \tau_{N+1}]$ for each of which is assumed $d_k^N = d_k^{0N} + x_k^N$. Here $d_k^{0N}$ is the zero approximation, while $x_k^N$ is a small correction. Let us introduce the vectors

$$x_N=\{x_1^N, \ldots, x_{15}^N\}^\tau, \quad d=\{d_1, \ldots, d_{15}\}^\tau,$$
$$d_0=\{d_1^0, \ldots, d_{15}^0\}^\tau.$$

In each $[\tau_N, \tau_{N+1}]$ we find that

$$\frac{dx_N}{dt}=0. \tag{4.2}$$

Equation (4.2) corresponds to (2.1), but since the state does not change, we have the case examined in the preceeding section.

Without making any assumptions on the interrelationship of estimates in each of the subdivision intervals, we will find by analogy in $[\tau_{N+1}, \tau_{N+2}]$ that

$$\frac{dx_{N+1}}{dt}=0. \tag{4.3}$$

Thus, equation (3.5) has the form

$$x_{N+1}=x_N. \tag{4.4}$$

Let us introduce the vector of discrepancies $\eta = \{\eta_x, \eta_y, \eta_z\}^T$ with the aid of the relationship

$$\begin{pmatrix}\eta_x\\ \eta_y\\ \eta_z\end{pmatrix}=\begin{pmatrix}H_x\\ H_y\\ H_z\end{pmatrix}-\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}H_{x_0}\\ H_{y_0}\\ H_{z_0}\end{pmatrix}. \tag{4.5}$$

It is not difficult to see that $\eta = \eta(t,d)$. By performing a linearization of discrepancies (4.5) in the neighborhood of the zero approximation $d_0$, we will find that

$$z=Hx_N+v. \tag{4.6}$$

It is designated in equation (4.6) that /163

$$z=\eta(t,\ d_0),\quad H=\frac{\partial\eta(t,\ d_0)}{\partial d_0}.$$

By performing discretization of system (4.6) for each $t_0^N, \ldots, t_q^N$, we will find that

$$\begin{gathered}z(t_0^N)=H(t_0^N)\,x_N+v(t_0^N),\\ \cdots\cdots\cdots\cdots\\ z(t_q^N)=N(t_q^N)\,x_N+v(t_q^N).\end{gathered} \tag{4.7}$$

By unifying systems (4.7), we find the analog of system (3.2)

$$z_N=H_Nx_N+v_N. \tag{4.8}$$

For system (4.8), relations (2.18) have the form

$$\begin{gathered}\hat{x}_N=\hat{x}_{N-1}+K_N(z_N-H_N\hat{x}_{N-1}),\\ K_N=P_NH_N^T[H_NP_NH_N^T+R_N]^{-1},\\ P_N=P_{N-1}-K_NH_NP_{N-1},\end{gathered} \tag{4.9}$$

and relations (2.19) will accordingly be

$$\hat{x}_N = \hat{x}_{N-1} + K_N (z_N - H_N \hat{x}_{N-1}), \tag{4.10}$$

$$\begin{aligned} K_N &= P_N H_N^T R_N^{-1}, \\ P_N^{-1} &= P_{N-1}^{-1} + H_N^T R_N^{-1} H_N. \end{aligned} \tag{4.11}$$

If in formula (4.10) we ignore the a priori information, i.e., posit $x_0 = 0$, $P_0^{-1} = 0$, then we will find an estimate of the type of least squares at the first step

$$\hat{x}_1 = [H_1^T R_1^{-1} H_1]^{-1} H_1^T R_1^{-1} z_1. \tag{4.12}$$

Relation (4.11) will be written now in the form

$$P_N^{-1} = \sum_{k=1}^{N} H_k^T R_k^{-1} H_k. \tag{4.13}$$

We will yield from (4.10)

$$\hat{x}_N = (I - K_N H_N)\, \hat{x}_{N-1} + P_N H_N^T R_N^{-1} z_N. \tag{4.14}$$

By virtue of (4.9), $P_N P_N^{-1} = I - K_N H_N$ , hence (4.14) will be written in the form

$$\hat{x}_N = P_N [P_{N-1}^{-1} \hat{x}_{N-1} + H_N^T R_N^{-1} z_N]. \tag{4.15}$$

By multiplying (4.15) on the left by $P_N^{-1}$, we yield

$$P_N^{-1} \hat{x}_N = P_{N-1}^{-1} \hat{x}_{N-1} + H_N^T R_N^{-1} z_N. \tag{4.16}$$

Since it follows from (4.12) that

$$P_1^{-1} \hat{x}_1 = H_1^T R_1^{-1} z_1,$$

then it follows from (4.16) that /164

$$P_{N-1}^{-1}\hat{x}_{N-1}=\sum_{k=1}^{N-1}H_k^{\mathrm{T}}R_k^{-1}z_k,$$

and then (4.15), allowing for (4.13), will be written as

$$\hat{x}_N=\left[\sum_{k=1}^{N}H_k^{\mathrm{T}}R_k^{-1}H_k\right]^{-1}\left[\sum_{k=1}^{N}H_k^{\mathrm{T}}R_k^{-1}z_k\right]. \tag{4.17}$$

The latter relation shows that in the case in point, filtration formulas in the calculating relation represent a multiple application of estimates of the method of least squares type.

Let us consider the continuous estimate of state for one and the same zero approximation $d_0$ for $[t_0, t_1]$.

Relations (2.20)-(2.21) will be written here as

$$\begin{aligned}\frac{d\hat{x}_t}{dt}&=K[z_t-H\hat{x}_t],\\ K&=PH^{\mathrm{T}}R^{-1},\\ \frac{dP}{dt}&=-PH^{\mathrm{T}}R^{-1}HP.\end{aligned} \tag{4.18}$$

By virtue of (2.31) we find that

$$d\hat{x}_t=PH^{\mathrm{T}}[GG^{\mathrm{T}}]^{-1/2}du. \tag{4.19}$$

It follows from (4.19) that the estimate of state is a process with independent increments, having a zero average and covariation matrix

$$E[d\hat{x}_t d\hat{x}_t^{\mathrm{T}}]=PH^{\mathrm{T}}R^{-1}HP\,dt.$$

Thus (4.19) yields the qualitative characteristics of trajectories of system (4.18).

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